International Journal of Theoretical Physics, Vol. 44, No. 2, February 2005 (© 2005)
DOI: 10.1007/s10773-005-1691-z

Dark Multi-Soliton Solution of the Nonlinear Schrödinger Equation with Non-Vanishing Boundary

Hao Cai,[1,2] Feng-Ming Liu,[1] and Nian-Ning Huang[1]

Received August 2, 2004; accepted November 24, 2004

The inverse scattering transform for the nonlinear Schrödinger equation in normal dispersion with non-vanishing boundary values is re-examined using an affine parameter to avoid double-valued functions. An operable algebraic procedure is developed to evaluate dark multi-soliton solutions. The dark two-soliton solution is given explicitly as an example, and is verified by direct substitution. The additional motion of the soliton center is given by its asymptotic behavior.

KEY WORDS: dark soliton; nonlinear equation; multi-soliton solution.

1. INTRODUCTION

The nonlinear Schrödinger equation in normal dispersion with non-vanishing boundary (simply NLS^+ equation) was solved by Zakharov and Sabat (1973), and a particular type of solution called dark soliton solution was obtained. While the single dark soliton solution was already given explicitly, the attempt to find the expression of multi-soliton solution was too onerous to be done (Zakharov and Shabat, 1972; Faddeev and Takhtajan, 1987). But the accurate expression of dark multi-soliton solution are basic to construct a general perturbation theory for dark solitons (Keener and McLaughlin, 1977; Kivshar and Malomad, 1989; Kaup and Newell, 1978; Huang *et al.*, 1999; Chen *et al.*, 1998). In the work of Zakharov and Sabat (1973), an affine parameter ζ was introduced as an auxiliary parameter to avoid double-valued function of original parameter and simplify the evaluation. The following theory should be developed in this way.

In this work, a systematic procedure is proposed to evaluate the dark multi-soliton solutions based upon the well-known linear algebraic formulae. And the dark two-soliton solution is given explicitly and the result is finally verified by

[1] Department of Physics, Wuhan University, Wuhan, People's Republic of China.
[2] To whom correspondence should be addressed at Department of Physics, Wuhan University, Wuhan 430072, People's Republic of China; e-mail: caihao2001@hotmail.com.

direct substitution. At the end of this work, the asymptotic behavior of this two-soliton solution is given and the effect between the two solitons composing this solution is found.

2. PRELIMINARY

The NLS$^+$ equation can be written as

$$iu_t - u_{xx} + 2(|u|^2 - \rho^2)u = 0 \tag{1}$$

with non-vanishing boundary conditions:

$$\begin{cases} u \to \rho & \text{as } x \to -\infty \\ u \to \rho e^{i\alpha} & \text{as } x \to \infty \end{cases} \tag{2}$$

where ρ is a positive constant. And its Lax pair is given by

$$L = -i\lambda\sigma_3 + U, \qquad U = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix} \tag{3}$$

and

$$M = i2\lambda^2\sigma_3 - 2\lambda U + i(U^2 - \rho^2 + U_x)\sigma_3 \tag{4}$$

In the limit of $x \to \infty$, the L tends to

$$L_+ = -i\lambda\sigma_3 + U_+ \tag{5}$$

where $U_+ = \rho\sigma_1$, and the corresponding free Jost solution is

$$E_+(x,\zeta) = (I + \rho\zeta^{-1}\sigma_2)e^{-i\kappa x} \tag{6}$$

where an auxiliary parameter ζ is introduced to avoid double-valued functions

$$\lambda = \frac{1}{2}(\zeta + \rho^2\zeta^{-1}), \qquad \kappa = \frac{1}{2}(\zeta - \rho^2\zeta^{-1}) \tag{7}$$

In the limit of $x \to -\infty$, the L tends to

$$L_- = -i\lambda\sigma_3 + U_- \tag{8}$$

where

$$U_- = Q(\alpha)U_+Q^{-1}(\alpha), \qquad Q(\alpha) = e^{-i\frac{1}{2}\alpha\sigma_3} \tag{9}$$

and the corresponding free Jost solution is

$$E_- = Q^{-1}(\alpha)E(x,\zeta) \tag{10}$$

Then the Jost solutions are defined as

$$\Psi(x,\zeta) = (\tilde{\psi}(x,\zeta), \quad \psi(x,\zeta)) \to E_+(x,\zeta), \quad \text{as } x \to \infty \tag{11}$$

$$\Phi(x,\zeta) = (\phi(x,\zeta), \quad \tilde{\phi}(x,\zeta)) \to E_-(x,\zeta), \quad \text{as } x \to -\infty \tag{12}$$

As usual, the monodramy matrix $T(\zeta)$ is introduced

$$\Phi(x,\zeta) = \Psi(x,\zeta)T(\zeta), \qquad T(\zeta) = \begin{pmatrix} a(\zeta) & \tilde{b}(\zeta) \\ b(\zeta) & \tilde{a}(\zeta) \end{pmatrix} \tag{13}$$

Since

$$\kappa > 0 \quad \text{if and only if} \quad \text{Im}\,\zeta > 0 \tag{14}$$

$\psi(x,\zeta)$, $\phi(x,\zeta)$ and $a(\zeta)$ are analytic in the upper half plane of complex ζ; $\tilde{\psi}(x,\zeta)$, $\tilde{\phi}(x,\zeta)$ and $\tilde{a}(\zeta)$ are analytic in the lower half plane of complex ζ. Usually $b(\zeta)$ and $\tilde{b}(\zeta)$ cannot be analytically continued outside the real axis.

The Jost solutions in NLS$^+$ equation have some properties, such as

$$\tilde{\psi}(x,\bar{\zeta}) = \sigma_1\overline{\phi(x,\zeta)}, \qquad \tilde{\phi}(x,\bar{\zeta}) = \sigma_1\overline{\phi(x,\zeta)} \tag{15}$$

and

$$\tilde{a}(\bar{\zeta}) = \overline{a(\zeta)}, \qquad \tilde{b}(\zeta) = \overline{b(\zeta)} \tag{16}$$

As a single value of λ results two values of ζ, there are $\lambda \to \lambda$ and $\kappa \to -\kappa$ under the so-called reduction transformation $\zeta \to \rho^2\zeta^{-1}$. Since $Q(\alpha)$ is un-effected by such a transformation, the Jost solutions have the following properties

$$\tilde{\psi}(x,\rho^2\zeta^{-1}) = i\rho^{-1}\zeta\psi(x,\zeta), \qquad \tilde{\phi}(x,\rho^2\zeta^{-1}) = -i\rho^{-1}\zeta\phi(x,\zeta) \tag{17}$$

We thus compare

$$\begin{aligned} \phi(x,\zeta) &= \tilde{\psi}(x,\zeta)a(\zeta) + \psi(x,\zeta)b(\zeta), \\ \tilde{\phi}(x,\zeta) &= \tilde{\psi}(x,\zeta)\tilde{b}(\zeta) + \psi(x,\zeta)\tilde{a}(\zeta) \end{aligned} \tag{18}$$

and

$$\begin{aligned} \phi(x,\rho^2\zeta^{-1}) &= \tilde{\psi}(x,\rho^2\zeta^{-1})\, a(\rho^2\zeta^{-1}) + \psi(x,\rho^2\zeta^{-1})\, b(\rho^2\zeta^{-1}) \\ \tilde{\phi}(x,\rho^2\zeta^{-1}) &= \tilde{\psi}(x,\rho^2\zeta^{-1})\, \tilde{b}(\rho^2\zeta^{-1}) + \psi(x,\rho^2\zeta^{-1})\, \tilde{a}(\rho^2\zeta^{-1}) \end{aligned} \tag{19}$$

using (17) and the definition of $T(\zeta)$ in (13), and then obtain

$$\tilde{a}(\rho^2\zeta^{-1}) = a(\zeta), \qquad \tilde{b}(\rho^2\zeta^{-1}) = -b(\zeta) \tag{20}$$

The last ones of Equations (15) and (20) are valid only for real ζ.

The first one of Lax equations can be rewritten in the form

$$\hat{L}\Psi(x,\zeta) = \lambda\Psi(x,\zeta), \qquad \hat{L} = i\sigma_3\partial_x - i\sigma_3 U \tag{21}$$

Since $\hat{L}$ is Hermitian operator, its eigenvalue λ must be real. From the relationship between ζ and λ, the discreet value ζ_n must be located on a upper half circle with radius ρ centered at the origin, that is

$$\zeta_n = \rho e^{i\beta_n}, \qquad 0 < \beta_n < \pi \tag{22}$$

And one should see that

$$\bar{\zeta}_n = \rho^2 \zeta_n^{-1} \tag{23}$$

The discreet spectrum part of $a(\zeta)$ is

$$a(\zeta) = e^{i\frac{1}{2}\alpha} \prod_{n=1}^{N} \frac{\zeta - \zeta_n}{\zeta - \bar{\zeta}_n} \tag{24}$$

where $\alpha = -2\sum_{n=1}^{N} \beta_n$. At zeros of $a(\zeta)$, and of $\tilde{a}(\zeta)$, there are

$$\phi(x, \zeta_n) = b_n \psi(x, \zeta_n), \qquad \tilde{\phi}(x, \bar{\zeta}_m) = \tilde{b}_m \tilde{\psi}(x, \bar{\zeta}_m) \tag{25}$$

where $b_n = b(\zeta_n)$ and $\tilde{b}_m = \tilde{b}(\zeta_m)$, and then

$$c_n \equiv -\frac{b_n}{\dot{a}(\zeta_n)\zeta_n} > 0 \tag{26}$$

which should be indicated in Appendix A.

The inverse scattering transform in the reflectionless case is

$$\tilde{\psi}(x, t, \zeta) = \left\{ \begin{pmatrix} 1 \\ i\rho\zeta^{-1} \end{pmatrix} - \sum_n \frac{1}{\zeta - \zeta_n} c_n \zeta_n \psi(x, t, \zeta_n) e^{i\kappa_n x} \right\} e^{-i\kappa x} \tag{27}$$

and the dark soliton solution is

$$\overline{u(x, t)} = \rho \left\{ 1 + \sum_n i c_n \rho^{-1} \zeta_n \psi_2(x, \zeta_n) e^{i\kappa_n x} \right\} \tag{28}$$

in which the time dependence relation derived from the second Lax equation of (3) is included, that is

$$b_n(t) = b_n(0) e^{-i4\kappa_n\lambda_n t}, \qquad c_n(t) = c_n(0) e^{-i4\kappa_n\lambda_n t}, \qquad r(t, \zeta) = r(0, \zeta) e^{-i4\kappa\lambda t} \tag{29}$$

3. DERIVATION OF MULTI-SOLITON SOLUTION BY SIMPLE ALGEBRAIC CALCULATION

Setting $\zeta = \bar{\zeta}_m$ in Equation (27) and then considering (17), we have

$$i\rho^{-1}\zeta_m \psi_2(x, \zeta_m) = i\rho^{-1}\zeta_m e^{i\kappa_m x} - \sum_n \frac{1}{\bar{\zeta}_m - \zeta_n} c_n \zeta_n \psi(x, t, \zeta_n) e^{i(\kappa_n + \kappa_m)x} \tag{30}$$

Introducing

$$\Psi_n = i\sqrt{c_n}\rho^{-1}\zeta_n \psi_2(x, \zeta_n), \qquad f_n = \sqrt{c_n} e^{i\kappa_n x}, \qquad g_n = i\rho^{-1}\zeta_n f_n \tag{31}$$

Equation (30) is rewritten in the matrix form

$$\Psi = g - \Psi B, \qquad \Psi(\Psi_1, \Psi_2, \ldots, \Psi_N), \qquad g = (g_1, g_2, \ldots, g_N) \tag{32}$$

and

$$B_{nm} = f_n \frac{\rho}{i(\bar{\zeta}_m - \zeta_n)} f_m \tag{33}$$

Then Equation (28) turns to

$$\bar{u} = \rho\{1 + \Psi f^T\} \tag{34}$$

Since

$$\Psi f^T = g(I + B)^{-1} f^T = \frac{\det(I + B + f^T g)}{\det(I + B)} - 1 \tag{35}$$

there is

$$\bar{u} = \rho \frac{\det(I + B')}{\det(I + B)} \tag{36}$$

in which

$$B' = B + f^T g, \qquad B'_{nm} = B_{nm} + f_n g_m = f_n \frac{\rho^{-1}}{i(\zeta_n - \bar{\zeta}_m)} \zeta_n \zeta_m f_m \tag{37}$$

Because

$$\det(I + B) = 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < n_2 < \cdots < n_r \le N} B(n_1, n_2, \ldots, n_r) \tag{38}$$

where $B(n_1, n_2, \ldots, n_r)$ are principal minors, it is easily found

$$B(n_1, n_2, \ldots, n_r) = \prod_{m,n} f_n^2 [i(\bar{\zeta}_m - \zeta_n)]^{-1} \prod_{n<m} [i(\bar{\zeta}_m - \bar{\zeta}_n)][i(\zeta_m - \zeta_n)] \rho^r \tag{39}$$

where $n, m \in \{n_1, n_2, \ldots, n_r\}$, and then there is

$$B(n_1, n_2, \ldots, n_r) = \prod_{n} \frac{\rho}{i(\bar{\zeta}_n - \zeta_n)} f_n^2 \prod_{n<m} \left| \frac{\zeta_n - \zeta_m}{\bar{\zeta}_n - \zeta_m} \right|^2 \tag{40}$$

Similarly, we can obtain the explicit expression of $\det(I + B')$

$$\det(I + B') = 1 + \sum_{r=1}^{N} \sum_{1 \le n_1 < n_2 < \cdots < n_r \le N} B'(n_1, n_2, \ldots, n_r) \tag{41}$$

where $B'(n_1, n_2, \ldots, n_r)$ are principal minors,

$$B'(n_1, n_2, \ldots, n_r) = \prod_{m,n} f_n^2 \zeta_n^2 [i(\bar{\zeta}_m - \zeta_n)]^{-1} \prod_{n<m} [i(\bar{\zeta}_m - \bar{\zeta}_n][i(\zeta_m - \zeta_n)] \rho^{-r} \tag{42}$$

or

$$B'(n_1, n_2, \ldots, n_r) = \prod_n \frac{\rho}{i(\bar{\zeta}_n - \zeta_n)} f_n^2 \rho^{-2} \zeta_n^2 \prod_{n<m} \left| \frac{\zeta_n - \zeta_m}{\bar{\zeta}_n - \zeta_m} \right|^2 \tag{43}$$

where $\rho^{-2}\zeta_n^2 = e^{i2\beta_n}$ and $n, m \in \{n_1, n_2, \ldots, n_r\}$. Substituting them into Equation (36), we finally obtain the expression of dark N-soliton solution.

4. EXPLICIT EXPRESSION OF DARK TWO-SOLITON SOLUTION

In the simplest case of $N = 1$, we have

$$\det(I + B) = 1 + i\rho f_1^2 \frac{1}{\zeta_1 - \bar{\zeta}_1}, \quad \det(I + B') = 1 + i\rho^{-1} f_1^2 \frac{1}{\zeta_1 - \bar{\zeta}_1} \zeta_1^2 \tag{44}$$

and then introduce h_1^2

$$f_1^2 = e^{i2\kappa_1 x} \frac{b_1}{\dot{a}(\zeta_1)\zeta_1} = h_1^2 \frac{\zeta_1 - \bar{\zeta}_1}{i\rho} \tag{45}$$

Since $\frac{\zeta_1 - \bar{\zeta}_1}{i\rho}$ is real, there is

$$h_1^2 = e^{-2\theta_1}, \qquad \theta_1 = k_1(x + 2\lambda_1 t) - k_1 x_1 \tag{46}$$

where $\kappa_1 = ik_1$, $\lambda_1 = \frac{1}{2}(\zeta_1 + \rho^2 \zeta_1^{-1})$ is real. Finally, we obtain

$$\bar{u}_1 = \rho \frac{\det(I + B')}{\det(I + B)} = \rho \frac{1 + e^{-2\theta_1} e^{-i2\beta_1}}{1 + e^{-2\theta_1}} \tag{47}$$

which is a well-known result and could be verified by direct substitution simply (Zakharov and Shabat, 1973).

For the dark two-soliton solution, i.e. $N = 2$, there is

$$\det(I + B) = 1 + i\rho \left\{ f_1^2 \frac{1}{\zeta_1 - \bar{\zeta}_1} + f_2^2 \frac{1}{\zeta_2 - \bar{\zeta}_2} \right\} - \rho^2 f_1^2 f_2^2 \frac{(\zeta_1 - \zeta_2)(\bar{\zeta}_1 - \bar{\zeta}_2)}{(\zeta_1 - \bar{\zeta}_1)(\zeta_2 - \bar{\zeta}_2)(\zeta_1 - \bar{\zeta}_2)(\zeta_2 - \bar{\zeta}_1)} \tag{48}$$

Similarly, h_1^2 and h_2^2 are introduced as

$$f_1^2 = h_1^2 \frac{\zeta_2(\zeta_1 - \bar{\zeta}_1)(\zeta_1 - \bar{\zeta}_2)}{i(\zeta_1 - \zeta_2)\rho^2}, \qquad f_2^2 = h_2^2 \frac{\zeta_1(\zeta_2 - \bar{\zeta}_2)(\zeta_2 - \bar{\zeta}_1)}{i(\zeta_2 - \zeta_1)\rho^2} \tag{49}$$

where $\frac{\zeta_2(\zeta_1 - \bar{\zeta}_1)(\zeta_1 - \bar{\zeta}_2)}{i(\zeta_1 - \zeta_2)\rho^2}$ and $\frac{\zeta_1(\zeta_2 - \bar{\zeta}_2)(\zeta_2 - \bar{\zeta}_1)}{i(\zeta_2 - \zeta_1)\rho^2}$ are real. As a result, the last term of Equation (48) is

$$-\rho^2 f_1^2 f_2^2 \frac{(\zeta_1 - \zeta_2)(\bar{\zeta}_1 - \bar{\zeta}_2)}{(\zeta_1 - \bar{\zeta}_1)(\zeta_2 - \bar{\zeta}_2)(\zeta_1 - \bar{\zeta}_2)(\zeta_2 - \bar{\zeta}_1)} = h_1^2 h_2^2 \tag{50}$$

The other two terms of Equation (41) are

$$i\rho f_1^2 \frac{1}{\zeta_1 - \bar{\zeta}_1} = h_1^2 \frac{\zeta_2}{\rho} \frac{(\zeta_1 - \bar{\zeta}_2)}{(\zeta_1 - \zeta_2)} = h_1^2 \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| \tag{51}$$

$$i\rho f_2^2 \frac{1}{\zeta_2 - \bar{\zeta}_2} = h_2^2 \frac{\zeta_1}{\rho} \frac{(\zeta_2 - \bar{\zeta}_1)}{(\zeta_2 - \zeta_1)} = h_2^2 \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| \tag{52}$$

Thus we have

$$\det(I + B) = 1 + (e^{-2\theta_1} + e^{-2\theta_2}) \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| + e^{-2(\theta_1 + \theta_2)} \tag{53}$$

Similarly,

$$\det(I + B') = 1 + (e^{-2\theta_1} e^{i2\beta_1} + e^{-2\theta_2} e^{i2\beta_2}) \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| + e^{-2(\theta_1 + \theta_2)} e^{i2(\beta_1 + \beta_2)} \tag{54}$$

Finally, a dark two-soliton solution is obtained by substituting all of them into Equation (36)

$$\bar{u}_2 = \rho \frac{1 + (e^{-2\theta_1} e^{i2\beta_1} + e^{-2\theta_2} e^{i2\beta_2}) \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| + e^{-2(\theta_1 + \theta_2)} e^{i2(\beta_1 + \beta_2)}}{1 + (e^{-2\theta_1} + e^{-2\theta_2}) \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| + e^{-2(\theta_1 + \theta_2)}} \tag{55}$$

which has been verified by direct substitution. There is the Figure 1 about the dark two-soliton solutions, in which the parameters are chosen as $\rho = 1$, $\beta_1 = \frac{\pi}{4}$, $\beta_2 = \frac{3\pi}{4}$ and $x_1 = x_2 = 0$.

5. ASYMPTOTIC BEHAVIOR

For the dark two-soliton solution, noticing λ_n is real, one could require

$$\lambda_2 < \lambda_1 \tag{56}$$

which means that the soliton corresponding to λ_2 is moving slower than that corresponding to λ_1. Due to Equation (22), there is then $0 < \beta_1 < \beta_2 < \pi$.

In order to find the effect from the faster-moving soliton on the slower-moving soliton, we discuss the neighborhood of the center of λ_2-soliton $x = x_2 + 2\lambda_2 t$, and then, in the limit of $t \to \infty$, there is

$$x - x_1 - 2\lambda_1 t \to -\infty \tag{57}$$

Since $k_n > 0$, we have

$$\theta_1 \to -\infty, \qquad e^{-2\theta_1} \to \infty \tag{58}$$

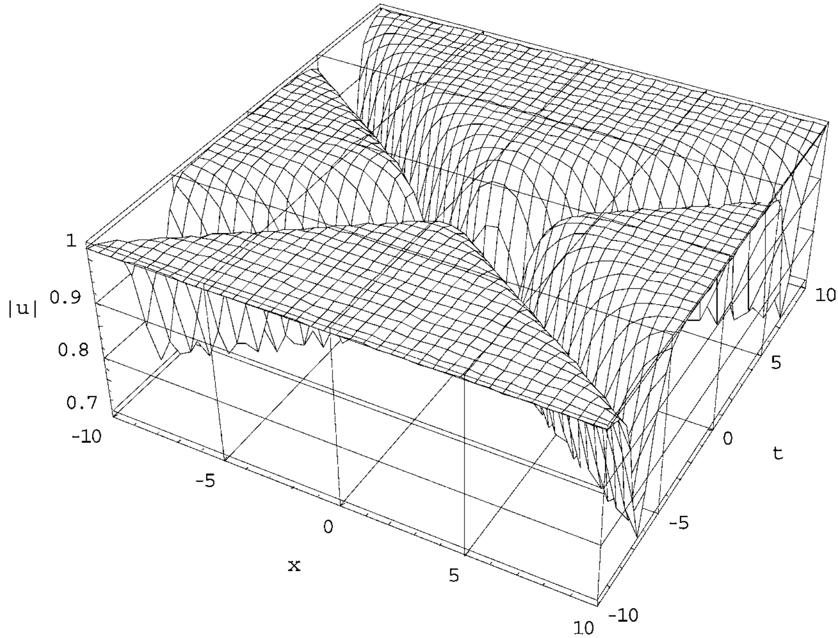

Fig. 1. The dark two-soliton solution of NLS$^+$ equation with parameters $\rho = 1, \beta_1 = \frac{\pi}{4}, \beta_2 = \frac{3\pi}{4}$ and $x_1 = x_2 = 0$.

that is

$$u_2 \cong \rho \frac{e^{-2\theta_1} e^{-i2\beta_1} \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| + e^{-2(\theta_1+\theta_2)} e^{-i2(\beta_1+\beta_2)}}{e^{-2\theta_1} \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| + e^{-2(\theta_1+\theta_2)}} \tag{59}$$

Introducing

$$\Delta_2 = \frac{1}{2k_2} \ln \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| \tag{60}$$

Equation (59) becomes

$$u_2 \cong \rho e^{-i2\beta_1} \frac{1 + e^{-2\theta_2^+} e^{-i2\beta_2}}{1 + e^{-2\theta_2^+}} \tag{61}$$

where

$$\theta_2^+ = \theta_2 + k_2 \Delta_2 = k_2 (x - x_2 - 2\lambda_2 t + \Delta_2) \tag{62}$$

On the other hand, in the limit of $t \to -\infty$, there is

$$x - x_1 - 2\lambda_1 t \to \infty, \qquad e^{-2\theta_1} \to 0 \tag{63}$$

and then

$$u_2 \cong \rho \frac{1 + e^{-2\theta_2^-} e^{-i2\beta_2}}{1 + e^{-2\theta_2^-}} \tag{64}$$

where

$$\theta_2^- = \theta_2 - k_2 \Delta_2 = k_2 (x - x_2 - 2\lambda_2 t - \Delta_2) \tag{65}$$

It should be noticed that (61) and (64) are similar to the expression of dark single-soliton solution (47). Since discreet value of ζ_n must be located on a upper half circle with radius ρ centered at the origin shown as in (22), there is

$$\left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| > 1, \qquad \Delta_2 > 0 \tag{66}$$

which means the center of the slower-moving λ_2-soliton moves additionally $2\Delta_2$ correspondingly to the faster-moving λ_1-soliton.

With similar procedure, we should find that the center of λ_1-soliton is also affected by λ_2 soliton. Introducing

$$\Delta_1 = \frac{1}{2k_1} \left| \frac{\zeta_1 - \bar{\zeta}_2}{\zeta_1 - \zeta_2} \right| > 0 \tag{67}$$

the center of λ_1-soliton moves additionally $-2\Delta_1$ correspondingly to the slower-moving λ_2-soliton.

ACKNOWLEDGMENTS

The project is supported by the National Science Foundation of China under Grant No. 10375041.

APPENDIX A

From Equation (13), we have

$$a(\zeta) = (1 - \rho^2 \zeta^{-2})^{-1} \det[\phi(x, \zeta), \psi(x, \zeta)] \tag{A.1}$$

and then

$$\left(1 - \rho^2 \zeta_n^{-2}\right) \dot{a}(\zeta_n) = \det[\dot{\phi}(x, \zeta_n), \psi(x, \zeta_n)] + \det[\phi(x, \zeta_n), \dot{\psi}(x, \zeta_n)] \tag{A.2}$$

noticing $\phi(x, \zeta_n) = b_n \psi(x, \zeta_n)$, that is

$$\left(1 - \rho^2 \zeta_n^{-2}\right) \dot{a}(\zeta_n) = b_n^{-1} \det[\dot{\phi}(x, \zeta_), \phi(x, \zeta_n)] + b_n \det[\psi(x, \zeta_n), \dot{\psi}(x, \zeta_n)] \tag{A.3}$$

From the first Lax equation, there are

$$\partial_x \phi(x, \zeta_n) = [-i\lambda_n \sigma_3 + U]\phi(x, \zeta_n) \tag{A.4}$$

$$\partial_x \dot{\phi}(x, \zeta_n) = [-i\dot{\lambda}_n \sigma_3]\phi(x, \zeta_n) + [-i\lambda_n \sigma_3 + U]\dot{\phi}(x, \zeta_n) \tag{A.5}$$

where $\dot{\lambda}_n = \frac{\partial \lambda}{\partial \zeta}|_{\zeta=\zeta_n}$. As a result, it is not difficult to see that

$$\det[\dot{\phi}(x, \zeta_n)_x, \phi(x, \zeta_n)] + \det[\dot{\phi}(x, \zeta_n), \phi(x, \zeta_n)_x] = -i\dot{\lambda}_1 2\phi(x, \zeta_n)_1 \phi(x, \zeta_n)_2 \tag{A.6}$$

where $\phi(x, \zeta_n) = (\phi(x, \zeta_n)_1 \phi(x, \zeta_n)_2)^T$, that is

$$\det[\dot{\phi}(x, \zeta_n), \phi(x, \zeta_n)] = -i\dot{\lambda}_n 2 \int_{-\infty}^{x} dx \phi(x, \zeta_n)_n \phi(x, \zeta_n)_2 \tag{A.7}$$

Similarly, we have

$$\det[\dot{\psi}(x, \zeta_n), \psi(x, \zeta_n)] = -i\dot{\lambda}_n 2 \int_{x}^{\infty} dx \psi(x, \zeta_1)_1 \psi(x, \zeta_n)_2 \tag{A.8}$$

Introducing (A7) and (A8) into (A6), we finally obtain

$$\dot{a}(\zeta_n) = -ib_n \int_{-\infty}^{\infty} \psi_1(x, \zeta_n)\psi_2(x, \zeta_n)\, dx \tag{A.9}$$

From Equations (15) and (17), there is

$$\psi_1(x, \zeta_n) = -i\rho\zeta_n^{-1}\overline{\psi_2(x, \zeta_n)} \tag{A.10}$$

that is

$$\dot{a}(\zeta_n) = b_n \rho \zeta_n^{-1} \int_{-\infty}^{\infty} |\psi_2(x, \zeta_n)|^2\, dx \tag{A.11}$$

Thus, we have

$$c_n \equiv -\frac{b_n}{\dot{a}(\zeta_n)\zeta_n} > 0. \tag{A.12}$$

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